UFR Sciences Amiens, License de Physique S5

Outils Mathématique pour le Physique : TD-5

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Exercice 1.

Calculer les imges de Fourier de fonctions suivantes

a)

$$f(t) = t^2 e^{-at^2} \quad (a > 0)$$

b)

$$f(t) = te^{-at^2} \quad (a > 0)$$

c)

$$f(t) = e^{-at^2} \cos \Omega t$$

d)

$$f(t) = e^{-|at|}$$

e)

$$f(t) = e^{-at^2} \sin \Omega t$$

f)

$$f(t) = \theta(|t| - \pi) \cos t$$

g)

$$f(t) = \frac{e^{-|at|}}{\sqrt{|t|}}$$

SOLUTION

$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-at^2}e^{-iqt}dt = \frac{1}{\sqrt{2\pi}}e^{-\frac{q^2}{4a}}\int_{-\infty}^{\infty} e^{-\left(\sqrt{a}t+\frac{iq}{2\sqrt{a}}\right)^2}dt = \frac{1}{\sqrt{2\pi}}\frac{1}{\sqrt{a}}e^{-\frac{q^2}{4a}}\sqrt{\pi} = \frac{1}{\sqrt{2a}}e^{-\frac{q^2}{4a}}$

a) $F(q) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} t^2 e^{-at^2}e^{-iqt}dt = -\frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial a}\int_{-\infty}^{\infty} e^{-at^2}e^{-iqt}dt = -\frac{1}{\sqrt{2\pi}}\frac{\partial}{\partial a}\frac{1}{\sqrt{a}}e^{-\frac{q^2}{4a}}\sqrt{\pi} = \frac{1}{(2a)^{3/2}}\left(1-\frac{1}{2}\frac{q^2}{a}\right)e^{-\frac{q^2}{4a}}$

b) $F(q) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} te^{-at^2}e^{-iqt}dt = \frac{1}{\sqrt{2\pi}}i\frac{\partial}{\partial q}\int_{-\infty}^{\infty} e^{-at^2}e^{-iqt}dt = \frac{1}{\sqrt{2\pi}}i\frac{\partial}{\partial q}\frac{1}{\sqrt{a}}e^{-\frac{q^2}{4a}}\sqrt{\pi} = \frac{-iq}{(2a)^{3/2}}e^{-\frac{q^2}{4a}}$

c) $F(q) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-at^2}\cos\Omega te^{-iqt}dt = \frac{1}{\sqrt{2\pi}}\frac{1}{2}\int_{-\infty}^{\infty} e^{-at^2}e^{-i(q+\Omega)t}dt + (\Omega \to -\Omega) = \frac{1}{2\sqrt{2a}}\left[e^{-\frac{(q+\Omega)^2}{4a}} + e^{-\frac{(q-\Omega)^2}{4a}}\right]$

d) $F(q) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-|at|}e^{-iqt}dt = \frac{1}{\sqrt{2\pi}}\mathrm{Re}2\int_{0}^{\infty} e^{-(|a|+iq)t}dt = \frac{1}{\sqrt{2\pi}}\mathrm{Re}\frac{2}{|a|+iq} = \frac{1}{\sqrt{2\pi}}\frac{2|a|}{a^2+q^2}$

e) $F(q) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-at^2}\sin\Omega te^{-iqt}dt = \frac{1}{\sqrt{2\pi}}\frac{1}{2i}\int_{-\infty}^{\infty} e^{-at^2}e^{-i(q-\Omega)t}dt - (\Omega \to -\Omega) = \frac{i}{2\sqrt{2a}}\left[e^{-\frac{(q+\Omega)^2}{4a}} - e^{-\frac{(q-\Omega)^2}{4a}}\right]$

f) $F(q) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\theta(|t| - \pi)\cos te^{-iqt}dt = \frac{1}{\sqrt{2\pi}}\int_{-\pi}^{\pi}\cos t\cos qtdt = \frac{1}{\sqrt{2\pi}}\left[\frac{\sin(1+q)\pi}{(1+q)\pi} + \frac{\sin(1-q)\pi}{(1-q)\pi}\right]$

g) $F(q) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{e^{-|at|}}{\sqrt{|t|}}e^{-iqt}dt = \frac{2}{\sqrt{2\pi}}\mathrm{Re}\int_{0}^{\infty}\frac{e^{-|a|t}}{\sqrt{t}}e^{-iqt}dt \overset{t=x^2}{=} \frac{4}{\sqrt{2\pi}}\mathrm{Re}\int_{0}^{\infty} e^{-(|a|+iq)x^2}dx = 2\sqrt{2}\mathrm{Re}\frac{1}{\sqrt{|a|+iq}}$